

Algebra II – Finding Rational Zeros

Objective: The student will be able to use the Rational Zero Theorem, the Remainder and Factor Theorems, synthetic division, the graphing calculator and the various methods of solving quadratic equations to find all rational zeros of a polynomial function.

Description: The students will use the explanation and examples that follow to practice finding rational zeros.

Preparation: The students should know:

- the Remainder and Factor Theorems;

- how to do synthetic division;

- how to solve quadratic equations by:

 - using special factoring patterns

 - by finding square roots

 - by completing the square

 - by using the quadratic formula

- how to graph a polynomial function on a graphing calculator

Arkansas Frameworks: Strand 5: Patterns, Algebra and Functions

PAF.1.1, PAF.1.3, PAF.2.3, PAF.2.5, PAF.2.7

Assessment: Successful completion of the following worksheet.

Finding Rational Zeros

Yesterday we used synthetic division to find factors of $f(x)$ when we know one zero, or solution. That's all well and good if you know one zero. But what if you don't?

That's where the Rational Zero Theorem comes in:

If $f(x) = a_n x^n + \dots + a_1 x^1 + a_0$ has integer coefficients, then every rational zero of f has the following form:

$$\frac{p}{q} = \frac{\text{factor of the constant term } a_0}{\text{factor of the leading term } a_n}$$

Example:

Find rational zeros of $f(x) = x^3 - 4x^2 - 11x + 30$

Constant term: 30 factors: 1,2,3,5,6,10,15,30

Leading coefficient: 1 factors: 1

Possible zeros: $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$

We'll test these zeros with synthetic division.

$$\begin{array}{r|rrrr} 1 & 1 & -4 & -11 & 30 \\ & & 1 & -3 & -14 \\ \hline & 1 & -3 & -14 & 16 \end{array}$$

$$\begin{array}{r|rrrr} -1 & 1 & -4 & -11 & 30 \\ & & -1 & 5 & 6 \\ \hline & 1 & -5 & -6 & 36 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & -4 & -11 & 30 \\ & & 2 & -4 & -30 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

The zero remainder shows us our factor.

The factorization is:

$$\begin{aligned} f(x) &= (x - 2)(x^2 - 2x - 15) \\ &= (x - 2)(x - 5)(x + 3) \end{aligned}$$

This trinomial can be factored further

Using the Zero Product Property, we get $x = 2, 5, -3$

Sometimes one round of synthetic division doesn't get our polynomial small enough to factor manually. In this case, we use synthetic division on the solution of the previous division, until we get something we can handle.

Find the rational zeros of $f(x) = 15x^4 - 68x^3 - 7x^2 + 24x - 4$

Constant term: -4 factors: $\pm 1, \pm 2, \pm 4$

Leading coefficient: 15 factors: $\pm 1, \pm 3, \pm 5, \pm 15$

Possible zeros: $\pm 1, \pm 2, \pm 4, \pm 1/3, \pm 2/3, \pm 4/3, \pm 1/5, \pm 2/5, \pm 4/5, \pm 1/15, \pm 2/15, \pm 4/15$

To narrow these possibilities down, I graphed the equation and looked for where the graph crossed the x-axis. $+2/3$ was a possibility, but $+1/5$ was a definite.

$$\begin{array}{r|rrrrr} 1/5 & 15 & -68 & -7 & 24 & -4 \\ & & 3 & -13 & -4 & 4 \\ \hline & 15 & -65 & -20 & 20 & 0 \end{array}$$

This is a good answer, so let's factor.

$$f(x) = (x - 1/5)(15x^3 - 65x^2 - 20x + 20)$$

There's a common factor here.

$$f(x) = (x - 1/5) 5(3x^3 - 13x^2 - 4x + 4)$$

Now we can use the five to get rid of the fraction

$$f(x) = (5x - 1)(3x^3 - 13x^2 - 4x + 4)$$

Now we do the process again on our new

polynomial.

Find the rational zeros of $g(x) = (3x^3 - 13x^2 - 4x + 4)$

Constant term: 4 factors: $\pm 1, \pm 2, \pm 4$

Leading coefficient: 3 factors: $\pm 1, \pm 3$

Possible zeros: $\pm 1, \pm 2, \pm 4, \pm 1/3, \pm 2/3, \pm 4/3$

Instead of checking each possibility, I graphed the function and did some tracing. $-2/3$ looked to be a good bet.

$$\begin{array}{r|rrrr} -2/3 & 3 & -13 & -4 & 4 \\ & \underline{-2} & \underline{10} & \underline{-4} & \\ & 3 & -15 & 6 & 0 \end{array}$$

This factorization is:

$$g(x) = (x + 2/3)(3x^2 - 15x + 6)$$

We've got a common factor again.

$$g(x) = (x + 2/3)3(x^2 - 5x + 2)$$

Get rid of the fraction

$$g(x) = (3x + 2)(x^2 - 5x + 2)$$

Now get back to the whole function

$f(x) = (5x - 1)(3x + 2)(x^2 - 5x + 2)$ This can't be factored any further, so we use the Zero Product Property to solve the first two terms, and the Quadratic Formula for the third term.

$$5x - 1 = 0 \quad * \quad 3x + 2 = 0 \quad * \quad x = (5 \pm \sqrt{((-5)^2 - 4(1)(2))}) / 2(1)$$

$$5x = 1 \quad * \quad 3x = -2 \quad * \quad x = (5 \pm \sqrt{(25 - 8)}) / 2$$

$$x = 1/5 \quad * \quad x = -2/3 \quad * \quad x = (5 + \sqrt{(17)}) / 2, (5 - \sqrt{(17)}) / 2$$

A rectangular column of cement is to have a volume of 20.25 cubic feet. The base is square, with sides 3 feet less than $1/2$ the height of the column. What should the dimensions of the column be?

First, sketch and label the column

Then, find the formula for the Volume of a rectangular solid: $V = lwh$

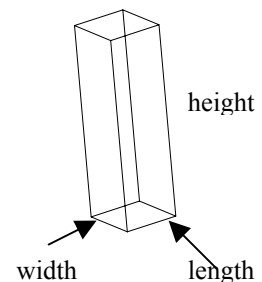
Now, substitute what we know into the formula.

$$\text{Volume} = 20.25 \text{ cubic feet}$$

$$\text{Length} = 1/2 \text{ height} - 3 \text{ feet}$$

$$\text{Width} = 1/2 \text{ height} - 3 \text{ feet}$$

$$\text{Height} = x$$



$$20.25 = (1/2x - 3)(1/2x - 3)x$$

Multiply the binomials

$$20.25 = (1/4x^2 - 3x + 9)x$$

Use the Distributive property to multiply by x

$$20.25 = 1/4x^3 - 3x^2 + 9x$$

Multiply everything by 4 to get rid of the fraction

$$81 = x^3 - 12x^2 + 36x$$

Subtract 81 from both sides to get standard form.

$$0 = x^3 - 12x^2 + 36x - 81$$

Now we start the process:

Constant term: -81 factors: $\pm (1, 3, 9, 27, 81)$

Leading coeff.: 1 factors: ± 1

Possible zeros: 1,3,9,27,81 (since we're looking for dimensions here, we can ignore the negative zeros)

$$\begin{array}{r|rrrr} 1 & 1 & -1 & 36 & -81 \\ & & 1 & -11 & 25 \\ \hline & 1 & -11 & 25 & -56 \end{array} \quad \begin{array}{r|rrrr} 3 & 1 & -12 & 36 & -81 \\ & & 3 & -27 & 27 \\ \hline & 1 & -9 & 9 & -34 \end{array} \quad \begin{array}{r|rrrr} 9 & 1 & -12 & 36 & -81 \\ & & 9 & -27 & 81 \\ \hline & 1 & -3 & 9 & 0 \end{array}$$

Eureka!!! Our zero is 9.

$0 = (x - 9)(x^2 - 3x + 9)$ The second term can't be factored, so we'll have to use the Quadratic formula on it.

$$0 = x - 9$$

$$9 = x \quad \text{Possible answer}$$

$$x = [3 \pm \sqrt{(-3)^2 - 4(1)(9)})] / 2(1)$$

$$x = [3 \pm \sqrt{9 - 36}] / 2$$

$x = [3 \pm \sqrt{-25}] / 2$ STOP!!!! This is going to be a complex number, and can't possibly be an answer to the height of a column, so ignore this one.

Our column height is 9 feet. We still need to know the length and width of the base, however.

$$\begin{aligned} \text{Length (and width, too, since the base is square)} &= \frac{1}{2}(9) - 3 \\ &= 4.5 - 3 \\ &= 1.5 \text{ feet} \end{aligned}$$

So the dimensions of our concrete column are 9 ft x 1.5 ft x 1.5 ft.

Worksheet
Finding Rational Zeros

Find all rational zeros of the function.

1. $f(x) = x^3 - 8x^2 - 23x + 30$

2. $f(x) = x^3 - 7x^2 + 2x + 40$

3. $f(x) = x^3 + x^2 - 2x - 2$

4. $f(x) = x^4 + x^3 + x^2 - 9x - 10$

5. $f(x) = x^5 - x^4 - 7x^3 + 11x^2 - 8x + 12$

6. $f(x) = 2x^3 + 4x - 2x - 4$

7. $f(x) = 3x^3 + 12x^2 + 3x - 18$

8. $f(x) = 2x^4 + x^3 - x^2 - x - 1$

9. $f(x) = 2x^5 + x^4 - 32x - 16$

10. At a factory, molten glass is poured into molds to make paperweights. Each mold is a rectangular prism whose height is 3 inches greater than the length of each side of the square base. A machine pours 20 cubic inches of liquid glass into each mold. What are the dimensions of the mold? (Volume of a rectangular prism is lwh , where l is length, w is width and h is height.)

Answer Key
Finding Rational Zeros

1. $-3, 1, 10$
2. $-2, 4, 5$
3. $\pm\sqrt{2}, -1$
4. $-1, 2$
5. $-1, 2$
6. $-2, -1, 1$
7. $-3, -2, 1$
8. $-1, 1$
9. $-2, -\frac{1}{2}, 2$
10. 2 inches by 2 inches by 5 inches